Numerical Solutions of a Projectile Motion Model

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Outline

• Introduction
• Defining and Solving the Problem
• Fixed Points and Iterative Methods
• Inverse and Optimization Problem
• Numerical Algorithms and Results
• Conclusion
In this work, we will explore the concept of inverse problems. An inverse problem is one where we know the effect and want to determine the cause. For example, in physics, we might know the response of a system to a given input and want to determine the properties of the system.

**Figure:** Inverse Problems

- What is an Inverse Problem?
  - Forward Problem
  - Model $m$
  - Data $d$
  - Inverse Problem
Inverse Problem

- What is an Inverse Problem?
- What do they Influence?

Figure: Inverse Problems
Inverse Problem

• What is an Inverse Problem?

• What do they Influence?

• In this work?

Figure: Inverse Problems
Techniques Needed

• Differential Equations
  — To build model representing projectile motion
• Fixed Points and Fixed Point Iteration
  — Numerically solve implicitly defined model
• Optimization
  — Optimize the possible range
• Numerical Methods
  — Solve inverse optimization problem numerically
Defining the Problem

• Suppose we launch a point projectile from the origin with
  — Initial angle $\theta$ (radians)
  — Initial velocity $v$ (feet/second)
  — Unit mass (1 gram)

• The projectile is then subject to
  — Air resistance with coefficient $k$
  — Gravitational force $g = -32 \, (ft/sec^2)$

• The total forces can thus be represented by

\[-k \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad (1)\]
Figure : Graph of Projectile Motion
We can develop a system of two initial value problems (IVPs) to represent the motion of the projectile.

\[
\begin{align*}
\ddot{x} &= -k \dot{x} \\
\dot{x}(0) &= v \cos \theta \\
x(0) &= 0 \\
\ddot{y} &= -k \dot{y} - g \\
\dot{y}(0) &= v \sin \theta \\
y(0) &= 0
\end{align*}
\]
Solving the Problem

- Solving the initial value problems through basic substitution methods, we reach

\[ x = \frac{v \cos \theta (1 - e^{-kt})}{k} \quad (4) \]

\[ y = \left( \frac{v \sin \theta}{k} + \frac{g}{k^2} \right) (1 - e^{-kt}) - \frac{g}{k} t \quad (5) \]
Solving the Problem Cont’d

- Solving (4) for $t$ we have,

$$ t = -\frac{1}{k} \ln \left( 1 - \frac{ks}{v \cos \theta} \right) \quad (6) $$

substituting (6) into (5) and simplifying we have

$$ y = x \left( \frac{v \sin \theta}{k} + \frac{g}{k^2} \right) \left( \frac{kx}{v \cos \theta} \right) + \frac{g}{k^2} \ln \left( 1 - \frac{kx}{v \cos \theta} \right) \quad (7) $$

Thus we know $x$ is a root of the equation (7). We then have,

$$ x = \frac{v \cos \theta}{k} \left( 1 - e^{-\left( \frac{k}{v} \sec \theta + \frac{k^2}{g} \tan \theta \right) x} \right) \quad (8) $$
Defining Range Function

- The range equals the distance moved in the x direction, thus we can see that $x = R(\theta)$ is a root of

$$R(\theta) = \frac{\cos \theta}{a} (1 - e^{-A(\theta)R(\theta)})$$

(9)
Defining Range Function

• The range equals the distance moved in the x direction, thus we can see that $x = R(\theta)$ is a root of

$$R(\theta) = \frac{\cos \theta}{a} (1 - e^{-A(\theta)R(\theta)}) \quad (9)$$

• where

$$A(\theta) = a \sec \theta + b \tan \theta$$

$$a = \frac{k}{v} \quad \text{and} \quad b = \frac{k^2}{g}, \quad a > 0, \ b > 0.$$
In (9) the range value, \( R(\theta) \), is defined implicitly. It can be written in equivalent functional form

\[
F(\theta, r) = \frac{\cos \theta}{a} \left( 1 - e^{A(\theta)r} \right), \quad r > 0 \text{ and } \theta \in \left[ 0, \frac{\pi}{2} \right]
\]

For future reference, note
- \( a, A(\theta) \) are as defined above
- \( \theta \in \left[ 0, \frac{\pi}{2} \right] \) implies \( \frac{\cos \theta}{a} > 0 \) and \( \frac{\cos \theta A(\theta)}{a} > 1 \)
- \( F_r(\theta, r) \) and \( F_\theta(\theta, r) \) exist and are continuous
- \( F(\theta, r) \) is classically differentiable and thus continuous on \( \left[ 0, \frac{\pi}{2} \right] \)
Fixed Points

Definition
A fixed point of a function \( f \) is defined as a point \( p \) such that \( f(p) = p \).

- Example: \( f(x) = x^2 \) has two fixed points \( x = 0 \) and \( x = 1 \).
- Graphically, fixed points of a function are intersections between that function and the line \( y = x \).

Figure: Graph of \( y = x^2 \) and \( y = x \)
• To study the fixed points of functional (10) we work with a simplified, but equivalent form. Let

\[ f(x) = C \left( 1 - e^{-dx} \right), \quad C > 0, \; Cd > 1, \; \& \; x > 0 \]  

(11)

where \( C = \frac{\cos\theta}{a} \) and \( d = A(\theta) \).
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It can easily be shown that
- 0 is a fixed point of \( f \), by definition
- For sufficiently small \( s \), \( f(s) > s \), proof using L’Hopital's Rule
- \( f(C) < C \) for \( C \) defined as above, from conditions on \( C \)
• Since $f$ is continuous, by the Intermediate Value Theorem, there exists a point, $p \in (0, C)$, such that $f(p) = p$. Thus, by definition, $p$ is a fixed point of $f$. 

• It is easily shown that the second derivative of $f$ is strictly negative and thus $f$ is concave down and thus the graph can intersect the line $y = x$ at a maximum of two points in the domain. Since 0 is a known fixed point, we conclude $p$ is a unique positive fixed point.

• Furthermore, it can be shown that if $f(x) > x$, then $x < p$ and consequently $f(x) < x \Rightarrow x > p$ for all $x \geq 0$. The proof of this follows from $p$ being unique.
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Furthermore, it can be shown that if $f(x) > x$, then $x < p$ and consequently $f(x) < x \iff x > p$ for all $x \geq 0$. The proof of this follows from $p$ being unique.
Iterative Methods

• It follows that for any $x \geq 0$ a sequence $\{x_{n+1} = f(x_n)\}$ will converge monotonically to $p$. Therefore, for any initial estimate, the sequence of fixed point iterations converges to the fixed point.
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The results found while studying fixed point iteration with equation (11) can be applied to (10). From this we conclude that $R(\theta)$ is the unique positive fixed point of $F(\theta, r)$ and the fixed point iteration is a suitable method of solving the implicitly defined functional in (9).
Inverse Problem

- We work with solving the inverse problem of finding the angle at which a projectile should be launched to reach a suboptimal range. We define

\[ g(t) = at = 1 + e^{-(at+b\sqrt{t^2-R(\theta)^2})} \]  

(12)

Note: \( R(\theta) \) is a solution of equation (10) if and only if \( t = \frac{\cos \theta}{a(1-e^{-A(\theta)R(\theta)})} \) is a root of the function \( g(t) \) defined in (12).

Proof:
\[ R(\theta) = \cos \theta a(1-e^{-A(\theta)R(\theta)}) = \frac{1}{a} \sec \theta \Rightarrow at = 1 - e^{-(at+b\tan \theta R(\theta))} = \Rightarrow g(t) = 0 \]  

(13)

The converse can be proved in similar fashion.
We work with solving the inverse problem of finding the angle at which a projectile should be launched to reach a suboptimal range. We define

\[
g(t) = at = 1 + e^{-(at+b\sqrt{t^2-R(\theta)^2})}
\]  

(12)

Note: \( R(\theta) \) is a solution of equation (10) if and only if \( t = R(\theta) \sec(\theta) \) is a root of the function \( g(t) \) defined in (12).

Proof:

\[
R(\theta) = \frac{\cos \theta}{a} (1 - e^{-A(\theta)R(\theta)})
\]

\[
\implies a \sec(\theta)R(\theta) = 1 - e^{-(a \sec \theta + b \tan \theta)R(\theta)}
\]

\[
\implies at = 1 - e^{-(at+b \tan \theta)R(\theta)}
\]

\[
\implies 0 = at - 1 + e^{-(at+b\sqrt{t^2-R(\theta)^2})}
\]

\[
\implies g(t) = 0
\]  

(13)

The converse can be proved in similar fashion.
We also develop the inverse problem of finding the angle corresponding to the maximum range. Note that the second partial derivative is negative, thus critical points are maximums.

\[
\frac{\cos \theta}{a} \left( A'(\theta)R(\theta) \right) e^{-A(\theta)R(\theta)} - \frac{\sin \theta}{a} \left( 1 - e^{-A(\theta)R(\theta)} \right) = 0
\]

\[
R(\theta) \left[ \tan \theta \left( e^{-A(\theta)R(\theta)} - 1 \right) + c \sec \theta e^{-A(\theta)R(\theta)} \right] = 0, \quad c = \frac{b}{a}
\]

\[
\sin \theta - \sin \theta e^{-A(\theta)R(\theta)} - c e^{-A(\theta)R(\theta)} = 0
\]

\[
\sin \theta = (\sin \theta + c)e^{-A(\theta)R(\theta)}
\] (14)
Optimization Cont’d

• Taking arc sine on both sides, which exists since \( \theta \in \left(0, \frac{\pi}{2}\right)\), we can find \( \theta \) the solution of the inverse problem. In order to compute the angle we must find an equivalent form that is suitably defined.

From (10) we can see

\[
e^{-A(\theta)R(\theta)} = 1 - a \sec \theta R(\theta)
\]  

(15)

Substituting (15) into (14) we have

\[
sin \theta = (\sin \theta + c)(1 - a \sec \theta R(\theta))
\]

\[\Rightarrow R(\theta) = \frac{(c/a) \cos \theta}{\sin \theta + c}\]

\[\Rightarrow A(\theta)R(\theta) = \frac{c + c^2 \sin \theta}{\sin \theta + c}\]

\[\Rightarrow \sin \theta = (\sin \theta + c)e^{-\left(\frac{c+c^2 \sin \theta}{\sin \theta+c}\right)}\]

(16)
• For the Direct Problem, we solve our implicitly defined equation (9) using the fixed point iteration method.
Numerical Algorithms

• For the Direct Problem, we solve our implicitly defined equation (9) using the fixed point iteration method.

• For the Inverse Problem, equation (16) can be written in the equivalent form

\[ x = e^{hx}, \quad x = \frac{e \sin \theta}{\sin \theta + c} \quad \& \quad h = \frac{1 - c^2}{e} \]  

(17)

The numerical algorithm then solves equation (17) using Newton’s Method, setting \( \sin \theta = \frac{cx}{e-x} \) and \( \theta = \sin^{-1} \left( \frac{cx}{e-x} \right) \).
Results — Direct Problem

- Solutions of the direct problem using fixed point iteration.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$R$</th>
<th>$V$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/12$</td>
<td>$70.88511102176$</td>
<td>100</td>
<td>$67.34060878040$</td>
</tr>
<tr>
<td>$2\pi/12$</td>
<td>$77.88306236704$</td>
<td>300</td>
<td>$2.12024170366 \times 10^2$</td>
</tr>
<tr>
<td>$3\pi/12$</td>
<td>$67.34060878040$</td>
<td>500</td>
<td>$3.53551174056 \times 10^2$</td>
</tr>
<tr>
<td>$4\pi/12$</td>
<td>$48.61653757497$</td>
<td>700</td>
<td>$4.94974708427 \times 10^2$</td>
</tr>
<tr>
<td>$5\pi/12$</td>
<td>$25.36860773980$</td>
<td>900</td>
<td>$6.36396102456 \times 10^2$</td>
</tr>
<tr>
<td>$6\pi/12$</td>
<td>$6.01470426990 \times 10^{-15}$</td>
<td>1000</td>
<td>$7.77817459295 \times 10^2$</td>
</tr>
</tbody>
</table>

Table: Values of range for varying values of speed and initial angle with fixed $k=1$
The range values computed numerically based on the direct problem.

Figure: Plot of theta vs. range for varying values of speed: \(v=100, 500, 1000\) and \(k=1\).
Results — Inverse Problem

- The following tables compare the angles for varying values of speed, which produce the maximum range.

<table>
<thead>
<tr>
<th>V</th>
<th>θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.4593625518009411</td>
</tr>
<tr>
<td>200</td>
<td>0.347971552133387</td>
</tr>
<tr>
<td>300</td>
<td>0.286456352026570</td>
</tr>
<tr>
<td>400</td>
<td>0.246237533256279</td>
</tr>
<tr>
<td>500</td>
<td>0.217435525427001</td>
</tr>
<tr>
<td>600</td>
<td>0.195582070744295</td>
</tr>
<tr>
<td>700</td>
<td>0.178320450757590</td>
</tr>
<tr>
<td>800</td>
<td>0.164274857263486</td>
</tr>
<tr>
<td>900</td>
<td>0.152581613689122</td>
</tr>
<tr>
<td>1000</td>
<td>0.142668003658631</td>
</tr>
</tbody>
</table>

Table: Angles which produce the optimum range for varying values of speed
Results — Inverse Problem Cont’d

• In the following figure, the speed starts at $V = 100$ ft./sec. and is incremented by 10. The graph plots the number of increments along the x-axis and the value of theta along the y-axis.

Figure: Increments of speed vs. value of theta that produces maximum range
Conclusion

- We modeled the range of a point projectile as a function of the angle of elevation based on scientific knowledge.
- We defined the initial conditions and equations of motion to reflect the air resistance on the projectile using trigonometry.
- We then studied fixed points and fixed point iteration, and used iterative methods to numerically solve the equation.
- We solved the inverse problem of finding the angle that produces either the maximum range or a given suboptimal range.
- We showed that the iteration sequence converges monotonically to the fixed point for any positive initial guess, this helps ensure numerical stability.
- We analyzed the relationship between the initial speed, the angle of elevation, and the range.
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Thank You

Questions?